PHYS 486 Problem Set #1

1 The Swap Operator

We have defined the swap operator as follows in class:

$$S_{ij} = n_i n_j + \overline{n}_i \overline{n}_j + X_i X_j (n_i \overline{n}_j + \overline{n}_i n_j)$$

Explicitly prove that $S_{ij}^2 = 1$ by taking the above expression and squaring it. In manipulating the resulting expression, you may use the following identities:

$$\begin{aligned} Xn &= \overline{n}X & X\overline{n} &= nX \\ n^2 &= n, & \overline{n}^2 &= \overline{n} \\ n\overline{n} &= 0, & \overline{n}n &= 0 \\ X^2 &= 1 & n + \overline{n} &= 1 \end{aligned}$$

In addition, keep in mind that operators acting on different bits (i.e., with different indices) commute.

2 The cNOT Operator

We can define the controlled-not (cNOT) operator as:

$$C_{ij} = \overline{n}_i + X_j n_i$$

Using the above definition of the controlled-not operator, show that:

$$S_{ij} = C_{ij} C_{ji} C_{ij}$$

Again, you should expand the product (which will contain a total of nine terms) and using the identities given in the first problem, show that it can be reduced to the definition of S_{ij}.

3 The Z Operator

We define the Z operator as:

$$Z = \overline{n} - n$$

Prove the following relations:

$$\begin{aligned} ZX + XZ &= 0 \\ Z^2 &= 1 \\ n &= \frac{1}{2}(1 - Z) \\ \overline{n} &= \frac{1}{2}(1 + Z) \\ C_{ij} &= \frac{1}{2}(1 + Z_i) + \frac{1}{2}X_j(1 - Z_i) \\ &= \frac{1}{2}(1 + X_j) + \frac{1}{2}Z_i(1 - X_j) \end{aligned}$$

4 The Hadamard Operator

Define the Hadamard operator H as:

$$H = \frac{1}{\sqrt{2}}(X + Z)$$

Prove the following relations:

$$\begin{aligned} HXH &= Z \\ HZH &= X \end{aligned}$$